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Total Energy of the Bianchi Type I Universes

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Using the symmetric energy-momentum complexes of Landau and Lifshitz, Papapetrou, and Weinberg, we obtain the energy of the universe in anisotropic Bianchi type I cosmological models. The energy (due to matter plus field) is found to be zero and this agrees with a previous result of Banerjee and Sen, who investigated this problem using the Einstein energy-momentum complex. Our result supports the importance of the energy-momentum complexes and contradicts the prevailing folklore that different energy-momentum complexes could give different and hence unacceptable energy distribution in a given space-time. The result that the total energy of the universe in these models is zero supports the viewpoint of Tryon. Rosen computed the total energy of the closed homogeneous isotropic universe and found it to be zero, which agrees with the studies of Tryon.

1. INTRODUCTION

The landmark observations of 2.7 K background radiation strongly support that some version of the big bang theory is correct. Tryon (1973) assumed that our universe appeared from nowhere about 10^{10} years ago and mentioned that the conventional laws of physics need not have been violated at the time of creation of the universe. He proposed that our universe must have a zero net value for all conserved quantities. He presented some arguments indicating that the net energy of our universe may be indeed zero. His big bang model (in which our universe is a fluctuation of the vacuum) predicted a homogeneous, isotropic, and closed universe consisting of matter and antimatter equally. Tryon (1973) also referred to an elegant topological argument by Bergmann that any closed universe has zero energy.

The subject of the energy of the universe remained in almost a slumbering state for a long period of time and was reopened by interesting work of

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Rosen (1994) and Cooperstock (1994). Rosen (1994) considered a closed homogeneous isotropic universe described by a Friedmann–Robertson–Walker (FRW) metric. He used the Einstein energy-momentum complex[2] and found that the total energy is zero. This result interested some general relativists, for instance, Johri *et al.* (1995) and Banerjee and Sen (1997). Johri *et al.* (1995), using the Landau and Lifshitz energy-momentum complex, showed that the total energy of an FRW spatially closed universe is zero at all times irrespective of the equations of state of the cosmic fluid. They also showed that the total energy enclosed within any finite volume of the spatially flat FRW universe is zero at all times.

The Bianchi type I solutions, under a special case, reduce to the spatially flat FRW solutions. Banerjee and Sen (1997), using the Einstein energy-momentum complex, studied the Bianchi type I solutions and found that the total (matter plus field) energy density is zero everywhere. As the spatially flat FRW solution is a special case of the Bianchi type I solutions, one observes that the energy-momentum complexes of Einstein and of Landau and Lifshitz give the same result for the spatially flat FRW solutions.

These results, though they appear to be very interesting, are usually not taken seriously because the use of energy-momentum complexes is restricted to Cartesian coordinates. There are many prescriptions for obtaining energy in a curved space-time, and some of them (quasilocal masses) are not limited to a particular coordinate system, whereas the (energy-momentum complexes) are restricted to the use of Cartesian coordinates. A large number of definitions of quasilocal mass (associated with a closed two-surface) have been proposed (Brown and York, 1993). Bergqvist (1992) studied several different definitions of quasilocal masses for the Kerr and Reissner–Nordström space-times and came to the conclusion that not even two of these definitions gave the same results. Contrary to this, in the last decade, several authors studied energy-momentum complexes and obtained stimulating results. We will discuss some of them in brief. The leading contributions of Virbhadra and his collaborators (Rosen, Parikh, Chamorro, and Aguirregabiria) have demonstrated with several examples that for a given space-time, different energy-momentum complexes give the same and acceptable energy distribution. Several energy-momentum complexes have been shown to give the same energy distribution for each of the following space-times: the Kerr–Newman space-time (Virbhadra 1990a,b. Cooperstock and Richardson, 1992), Vaidya space-time (Virbhadra, 1992), Einstein–Rosen space-time (Rosen and Virbhadra, 1993;

[2] To avoid any confusion, we mention that we use the term energy-momentum complex for one which satisfies the local conservation laws and gives the contribution from the matter (including all nongravitational fields) as well as the gravitational field. Rosen (1994) used the term pseudotensor for this purpose. We reserve the term energy-momentum pseudotensor for the part of the energy-momentum complex due to the gravitational field only.

Virbhadra, 1995), Bonnor–Vaidya space-time (Chamorro and Virbhadra, 1995), and all Kerr–Schild class space-times (Aguirregabiria *et al.*, 1996). Recently Virbhadra (1999) discussed that the concept of local or quasilocal mass is very useful in investigating the *Seifert conjecture* (Seifert, 1979) and the *hoop conjecture* (Thorne, 1972). He also showed that, for a general nonstatic, spherically symmetric space-time of the Kerr–Schild class, the Penrose quasilocal mass definition (Penrose, 1982) as well as several energy-momentum complexes yield the same results. For some other interesting papers on this subject see Virbhadra and Parikh (1993, 1994), Chamorro and Virbhadra (1996), and Xulu (1998a,b). We have already discussed that Banerjee and Sen (1997) studied the energy distribution with Bianchi type I metrics, using the Einstein definition. It is our present aim to investigate whether or not some other energy-momentum complexes yield the same results for the Bianchi type I metrics. We use the convention that Latin indices take values from 0 to 3 and Greek indices values from 1 to 3, and take the geometrized units $G = 1$ and $c = 1$

2. BIANCHI TYPE I SPACE-TIMES

The Bianchi type I space-times are expressed by the line element

$$ds^2 = dt^2 - e^{2l}dx^2 - e^{2m}\,dy^2 - e^{2n}\,dz^2 \tag{1}$$

where l, m, n are functions of t alone. The nonvanishing components of the energy-momentum tensor T_i^k [$\equiv 1/8\pi\ G_i^k$, where G_i^k is the Einstein tensor] are

$$\begin{aligned}
T_0^0 &= \frac{1}{8\pi}\,(\dot{l}\dot{m} + \dot{m}\dot{n} + \dot{n}\dot{l})\\
T_1^1 &= \frac{1}{8\pi}\,(\dot{m}^2 + \dot{n}^2 + \dot{m}\dot{n} + \ddot{m} + \ddot{n})\\
T_2^2 &= \frac{1}{8\pi}\,(\dot{n}^2 + \dot{l}^2 + \dot{n}\dot{l} + \ddot{n} + \ddot{l})\\
T_3^3 &= \frac{1}{8\pi}\,(\dot{l}^2 + \dot{m}^2 + \dot{l}\dot{m} + \ddot{l} + \ddot{m})
\end{aligned} \tag{2}$$

The dot over l, m, n stands for the derivative with respect to the coordinate t. The metric given by Eq. (1) reduces to the spatially flat Friedmann–Robertson–Walker metric as a special case. With $l(t) = m(t) = n(t)$, defining $R(t) = e^{l(t)}$ and transforming the line element (1) to t, x, y, z coordinates according to $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, and $z = r\cos\theta$ gives

$$ds^2 = dt^2 - [R(t)]^2\{dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)\} \tag{3}$$

which describes the well-known spatially flat Friedmann–Robertson–Walker space-time.

3. THE LANDAU AND LIFSHITZ ENERGY-MOMENTUM COMPLEX

The symmetric energy-momentum complex of Landau and Lifshitz (1987) is given by

$$L^{ij} = \frac{1}{16\pi} \mathcal{S}^{ijkl}_{\ \ ,kl} \tag{4}$$

where

$$\mathcal{S}^{ijkl} = -g(g^{ij}g^{kl} - g^{ik}g^{jl}) \tag{5}$$

L^{00} and $L^{\alpha 0}$ give the energy density and energy current (momentum) density components, respectively. The energy-momentum complex of Landau and Lifshitz (LL) satisfies the local conservation laws,

$$\frac{\partial L^{ik}}{\partial x^k} = 0 \tag{6}$$

where

$$L^{ik} = -g(T^{ik} + t^{ik}) \tag{7}$$

g is the determinant of the metric tensor g_{ik}, T^{ik} is the energy-momentum tensor of the matter and all nongravitational fields, and t^{ik} is known as the LL energy-momentum pseudotensor. Thus the locally conserved quantity L^{ik} contains contributions from the matter, nongravitational, and gravitational fields. For the expression for t^{ik} see Landau and Lifshitz (1987).

Integrating L^{ik} over the three-space gives the energy and momentum components

$$P^i = \int\int\int L^{i0} \, dx^1 \, dx^2 \, dx^3 \tag{8}$$

P^0 is the energy and P^α are momentum components. In order to calculate the energy and momentum density components of the line element (1), the required nonvanishing components of S^{ijkl} are

$$\mathcal{S}^{0101} = -e^{2m+2n}$$

$$\mathcal{S}^{0110} = e^{2m+2n}$$

$$\mathcal{S}^{0202} = -e^{2l+2n}$$
$$\mathcal{S}^{0220} = e^{2l+2n}$$
$$\mathcal{S}^{0303} = -e^{2l+2m}$$
$$\mathcal{S}^{0330} = e^{2l+2m} \tag{9}$$

Using the above results in (4) and (5), we obtain

$$L^{00} = L^{\alpha 0} = 0 \tag{10}$$

4. THE ENERGY-MOMENTUM COMPLEX OF PAPAPETROU

The symmetric energy-momentum complex of Papapetrou (1948) is given by

$$\Omega^{ij} = \frac{1}{16\pi} \mathcal{N}^{ijkl}_{,kl} \tag{11}$$

where

$$\mathcal{N}^{ijkl} = \sqrt{-g}(g^{ij}\eta^{kl} - g^{ik}\eta^{jl} + g^{kl}\eta^{ij} - g^{jl}\eta^{ik}) \tag{12}$$

and

$$\eta^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{13}$$

is the Minkowski metric. The energy-momentum complex of Papapetrou satisfies the local conservation laws

$$\frac{\partial \Omega^{ik}}{\partial x^k} = 0 \tag{14}$$

The locally conserved energy-momentum complex Ω^{ik} contains contributions from the matter, nongravitational, and gravitational fields. Ω^{00} and $\Omega^{\alpha 0}$ are the energy and momentum (energy current) density components. The energy and momentum are given by

$$P^i = \int\int\int \Omega^{i0}\, dx^1\, dx^2\, dx^3 \tag{15}$$

We wish to find the energy and momentum density components for the space-time described by the line element (1). The required nonvanishing components of $\mathcal{N}^{ijkl}$ are

$$\mathcal{N}^{0011} = -(1 + e^{-2l})\, e^{l+m+n}$$
$$\mathcal{N}^{0110} = e^{-l+m+n}$$
$$\mathcal{N}^{0022} = -(1 + e^{-2m})\, e^{l+m+n}$$
$$\mathcal{N}^{0220} = e^{l-m+n}$$
$$\mathcal{N}^{0033} = -(1 + e^{-2n})\, e^{l+m+n}$$
$$\mathcal{N}^{0330} = e^{l+m-n}$$
$$\mathcal{N}^{0101} = \mathcal{N}^{0202} = \mathcal{N}^{0303} = e^{l+m+n} \tag{16}$$

Using the above results in (11) and (12), we obtain

$$\Omega^{00} = \Omega^{\alpha 0} = 0 \tag{17}$$

5. THE WEINBERG ENERGY-MOMENTUM COMPLEX

The symmetric energy-momentum complex of Weinberg (1972) is given by

$$W^{ij} = \frac{1}{16\pi}\, \Delta^{ijk}_{,i} \tag{18}$$

where

$$\Delta^{ijk} = \frac{\partial h^a_a}{\partial x_i}\, \eta^{jk} - \frac{\partial h^a_a}{\partial x_j}\, \eta^{ik} - \frac{\partial h^{ai}}{\partial x^a}\, \eta^{jk} + \frac{\partial h^{aj}}{\partial x^a}\, \eta^{ik} + \frac{\partial h^{ik}}{\partial x_j} - \frac{\partial h^{jk}}{\partial x_i} \tag{19}$$

and

$$h_{ij} = g_{ij} - \eta_{ij} \tag{20}$$

η_{ij} is the Minkowski metric [see Eq. (13)]. The indices on h_{ij} or $\partial/\partial x_i$ are raised or lowered with the help of η's. The Weinberg energy-momentum complex W^{ik} contains contributions from the matter, nongravitational, and gravitational fields, and satisfies the local conservation laws

$$\frac{\partial W^{ik}}{\partial x^k} = 0 \tag{21}$$

W^{00} and $W^{\alpha 0}$ are the energy and momentum density components. The energy and momentum components are given by

$$P^i = \int\int\int W^{i0}\, dx^1\, dx^2\, dx^3 \tag{22}$$

We are interested in determining the energy and momentum density components. Now using Eqs. (1) and (19), we find that all the components of Δ^{ijk} vanish. Thus Eq. (18) yields

$$W^{ik} = 0 \tag{23}$$

6. DISCUSSION AND SUMMARY

The subject of energy-momentum localization in a curved space-time has been controversial. Misner *et al.* (1973) argued that the energy is localizable only for spherical systems. Cooperstock and Sarracino (1978) contradicted their viewpoint and argued that if the energy is localizable in spherical system, then it is localizable for all systems. Bondi (1990) advocated that a nonlocalizable form of energy is not admissible in relativity; therefore its location can in principle be found. The energy-momentum complexes are nontensorial under general coordinate transformations and are restricted to computations in Cartesian coordinates only. There has been a folklore that different energy-momentum complexes are very likely to give different and hence unacceptable energy distributions in a given space-time. To this end Virbhadra and coworkers and some others showed that several energy-momentum complexes "coincide" and give acceptable results for some well-known space-times. Their results influenced many researchers to work on this subject.

In recent years some researchers showed interest in studying the energy content of the universe in different models (Rosen, 1994; Cooperstock, 1994; Johri *et al.*, 1995; Banerjee and Sen, 1997). Rosen (1994), with the Einstein energy-momentum complex, studied the total energy of a closed homogeneous isotropic universe described by the Friedmann-Robertson–Walker (FRW) metric and found it to be zero. Using the Landau and Lifshitz definition of energy, Johri *et al.* (1995) demonstrated that (a) the total energy of an FRW spatially closed universe is zero at all times irrespective of the equations of state of the cosmic fluid and (b) the total energy enclosed within any finite volume of the spatially flat FRW universe is zero at all times. Banerjee and Sen (1997) showed that the energy and momentum density components vanish in the Bianchi type I space-times (they used the energy-momentum complex of Einstein).

It is usually suspected that different energy-momentum complexes could give different results for a given geometry. Therefore, we extended the investigations of Banerjee and Sen with three more energy-momentum complexes (proposed by Landau and Lifshitz, Papapetrou, and Weinberg) and found the same results [see Eqs. (10), (17), and (23)] as reported by them. Note that the energy density component of the energy-momentum tensor is not zero for the Bianchi type I solutions [see Eq. (2)]; however, it is clear from Eqs (10), (17), and (23) that the total energy density (due to matter plus field, as given by the energy-momentum complexes) vanishes everywhere. This is because the energy contributions from the matter and field inside an arbitrary two-surface in Bianchi type I space-times cancel each other. The results in this paper advocate the importance of energy-momentum complexes (opposes the folklore against them that different complexes could give different meaningless results for a given metric) and also supports the viewpoint of Tryon.

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